

Degenerate Perverse Sheaves on Abelian Varieties

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1 Relative generic vanishing

Let X be a complex abelian variety. Our aim is to show that an irreducible perverse sheaf on X with Euler characteristic zero is translation invariant with respect to some abelian subvariety of X of dimension > 0 .

Notation. Let $E(X)$ denote the perverse sheaves whose irreducible constituents K satisfy $\chi(K) = 0$. Let $N(X)$ denote *negligible* perverse sheaves, i.e. those for which all irreducible constituents are *translation invariant* for certain abelian subvarieties of X of dimension > 0 . Typical examples for translation invariant irreducible perverse sheaves are $\delta_X^\psi = L_\psi[\dim(X)]$, where L_ψ is the local system on X defined by a character $\psi : \pi_1(X, 0) \rightarrow \mathbb{C}^*$ of the fundamental group of X . Then $T_x^*(\delta_X^\psi) \cong \delta_x^\psi$ holds for all $x \in X$ and $\delta_x^\psi \in N(X)$. Let $F(X)$ denote the set of irreducible perverse sheaves in $E(X) \setminus N(X)$ up to isomorphism. For an arbitrary perverse sheaf K on X also its character twist $K_\psi = K \otimes_{\mathbb{C}_X} L_\psi$ is a perverse sheaf, and $N(X)$ and $E(X)$ are stable under twisting with ψ in this sense. Depending on the situation we sometimes write K^ψ instead of K_ψ for convenience, e.g. in the cases $\delta_X^\psi = (\delta_X)_\psi$. Let $M(X)$ denote perverse sheaves whose irreducible components M have Euler characteristic $\chi(M) \neq 0$.

For isogenies $f : X \rightarrow Y$ the functors f_* and f^* preserve $F(X)$, $E(X)$ and the categories of perverse sheaves; an easy consequence of the properties of the class N_{Euler} (defined in [KrW]) and the adjunction formula. A complex K is called *negligible*, if its perverse cohomology is in $N(X)$.

An irreducible perverse sheaf K on X is *maximal*, if for any quotient homomorphism $f : X \rightarrow B$ to a simple abelian quotient B and generic character twists K_χ of K the direct image $Rf_*(K_\chi)$ does not vanish. Let $F_{\max}(X)$ denote the maximal perverse sheaves in $F(X)$. By corollary 1 below (and the remark thereafter) for $K \in F_{\max}(X)$ one easily shows $Rf_*(K_{\chi_0}) \neq 0$ for any character χ_0 .

Our main result stated in theorem 4 is the assertion $F(X) = \emptyset$ respectively the equivalent

Theorem 1. *For an irreducible perverse sheaf K on a complex abelian variety X with vanishing Euler characteristic $\chi(K) = \sum_i (-1)^i \dim(H^i(X, K))$ there exists a nontrivial abelian subvariety $A \subseteq X$ such that $T_x^*(K) \cong K$ holds for all $x \in A$.*

For simple complex abelian varieties this is shown in [KrW]. The main result of this paper is the reduction of the theorem to the case of simple abelian varieties. We remark that, if we assume the corresponding result for simple abelian varieties over the algebraic closure k of a finite field κ , our proof of theorem 1 carries over to abelian varieties over k . In fact one step of our argument (in section 8) even uses methods of characteristic p referring to [W3, appendix]. In contrast, the proof for the case of simple complex abelian varieties used in [KrW] is of analytic nature, hence unfortunately can not be applied for fields of positive characteristic.

Reformulations of theorem 1. Simple perverse sheaves K on X are of the form $K = i_*(j_{!*}E_U)$ for some local system E_U on an open dense subvariety $j : U \hookrightarrow Z$ of the support $Z = \text{supp}(K)$. The support is an irreducible closed subvariety $i : Z \hookrightarrow X$ of X . If the irreducible perverse sheaf K is translation invariant with respect to an abelian subvariety A in X , its support Z satisfies $Z + A = Z$. By the Riemann-Hilbert correspondence there exists a regular singular holonomic D -module M on X , attached to K . The local system E_U defines an irreducible finite dimensional complex representation ϕ of the fundamental group $\pi_1(U)$ of U . By the A -invariance of the singular support of K there exists an A invariant closed subset Z' of Z , which contains the ramification locus of the perverse sheaf K . In other words, the restriction of the perverse K to $U = Z \setminus Z'$ is smooth in the sense that $K|_U = E[d]$ holds for a smooth etale sheaf E associated to a representation of the topological fundamental group $\pi_1(U)$ of U where U can be chosen such that $U + A = U$ holds.

Let \tilde{U} and \tilde{Z} denote the images of U and Z under the projection $q : X \rightarrow \tilde{X} = X/A$. Since $q : X \rightarrow \tilde{X} = X/A$ is smooth, by base change the induced morphism $q : Z \rightarrow \tilde{Z}$ is smooth. The smooth morphism $q : Z \rightarrow \tilde{Z}$ defines a Serre fibration $q : U \rightarrow \tilde{U}$. Since A is connected, we obtain from the long exact homotopy sequence

$$\pi_2(\tilde{U}) \xrightarrow{\delta} \pi_1(A) \xrightarrow{\sigma} \pi_1(U) \longrightarrow \pi_1(\tilde{U}) \longrightarrow 0 .$$

The first map δ in this sequence is zero, because $\pi_1(A)$ injects into $\pi_1(U)$. Indeed, consider the natural group homomorphism $\rho : \pi_1(U) \rightarrow \pi_1(X)$ induced from the inclusion $U \hookrightarrow X$. Obviously the composition $\rho \circ \sigma$ is the first map of the exact homology sequence

$$0 \longrightarrow \pi_1(A) \longrightarrow \pi_1(X) \longrightarrow \pi_1(\tilde{X}) \longrightarrow 0 .$$

hence $\rho \circ \sigma$, and therefore σ , is injective. To summarize: $\pi_1(A)$ is a normal subgroup of $\pi_1(U)$. We claim that $\pi_1(A)$ is in the center of $\pi_1(U)$. Indeed, for $\alpha \in \pi_1(A)$ and $\gamma \in \pi_1(U)$, there exists an $\alpha' \in \pi_1(A)$ such that $\gamma\alpha\gamma^{-1} = \alpha'$. If we apply the homomorphism ρ , this gives $\rho(\gamma)\rho(\alpha)\rho(\gamma)^{-1} = \rho(\alpha')$. Hence $\rho(\alpha) = \rho(\alpha')$, since $\pi_1(X) = H_1(X)$ is abelian. Therefore $\alpha = \alpha'$, because $\rho \circ \sigma$ is injective. Because $\pi_1(A)$ is a central subgroup of $\pi_1(U)$, for any irreducible representation ϕ of $\pi_1(U)$ there exists a character χ of $\pi_1(A)$ such that $\phi(\alpha\gamma) = \chi(\alpha)\phi(\gamma)$ holds for $\alpha \in \pi_1(A)$ and $\gamma \in \pi_1(U)$. Since $\pi_1(\tilde{X})$ is a free \mathbb{Z} -module, any character χ of $\pi_1(A)$ can be extended to a character χ_X of $\pi_1(X)$. Thus $\chi_X^{-1} \otimes \phi$ is an irreducible representation, which is trivial on $\pi_1(U)$; in other words it is an irreducible representation of \tilde{U} .

The last arguments imply that there exists a perverse sheaf \tilde{K} on \tilde{U} such that $L(\chi_X)^{-1} \otimes K = q^*(\tilde{K})[\dim(A)]$ holds on U . Then \tilde{K} necessarily is an irreducible perverse sheaf. Let \tilde{K} also denote the intermediate extension of \tilde{K} to \tilde{Z} , which is an irreducible perverse sheaf on \tilde{Z} . Since $q : Z \rightarrow \tilde{Z}$ is a smooth morphism with connected fibers, the pullback $q^*[\dim(A)]$ is a fully faithful functor from the category of perverse sheaves on \tilde{Z} to the category of perverse sheaves on Z . Also $L = q^*(\tilde{K})[\dim(A)]$ as perverse sheaf on Z is still irreducible on Z . Now K and L are both irreducible perverse sheaves on Z , whose restrictions on U coincide. Thus $K = L$. Choose a finite etale covering such that \tilde{X} splits. Then this implies (b) \implies (c) in the next theorem. The implications (b) \implies (c) \implies (a) are elementary, hence in view of theorem 1 we get

Theorem 2. *For an irreducible perverse sheaf K on a complex abelian variety X the following properties are equivalent*

- a) *The Euler characteristic $\chi(K)$ vanishes.*
- b) *There exists a positive dimensional abelian subvariety A of X , a translation invariant smooth sheaf $L(\chi_X)$ of rank one on X and a perverse sheaf \tilde{K} on $\tilde{X} = X/A$ such that $K \cong L(\chi_X) \otimes q^*(\tilde{K})[\dim(A)]$ holds for the quotient map $q : X \rightarrow \tilde{X}$.*

- c) *There exists a finite étale covering of X splitting into a product of two abelian varieties A and \tilde{X} , where $\dim(A) > 0$, such that the pullback of K is isomorphic to the external tensor product of a translation invariant perverse sheaf on A and a perverse sheaf on \tilde{X} .*

Outline of the proof of theorem 1. Any perverse sheaf K in $F(X)$ has an associated D -module whose characteristic variety as a subvariety of the cotangent bundle is a union of Lagrangians $\Lambda = \Lambda_Z$ for irreducible subvarieties $Z \subseteq X$. The assumption $\chi(K) = 0$ implies that all Z are degenerate. For this see [W] and [KrW]. For simple X therefore $Z = X$. By the Lagrangian property then Λ is the zero-section of the cotangent bundle $T^*(X)$. Hence by a well known theorem on D -modules K is attached to a local system defined and smooth on X , and then K is a translation invariant perverse sheaf on X . This proves the statement for simple abelian varieties and this essentially is the proof of [KrW]. The author's attempt to give a simple proof along these lines for general abelian varieties was not successful so far, and I would like to thank Christian Schnell for pointing out a gap.

For X isogenous to a product $A_1 \times A_2$ of two simple abelian varieties we use methods from characteristic p in the proof. We deal with this case in section 8 after some preparations in section 6 and 7 building on arguments that involve the tensor categories introduced in [KrW].

Finally when X has three or more simple factors, we simply use induction on $\dim(X)$. The main step, obtained in section 5, requires an analysis of the stalks of prime perverse sheaves in the sense of [W2]. These arguments are sheaf theoretic and use spectral sequences that naturally arise, if one restricts perverse sheaves on X to abelian subvarieties (e.g. fibers of homomorphisms used for the induction). This is described in section 2 and then is applied in section 4 and 5. A crucial step in section 4 is the reduction to the case of perverse sheaves in $F_{\max}(X)$.

An important tool for the study of homomorphisms $f : X \rightarrow B$ in the induction process will be the next lemma. This lemma can be easily derived from the special case of the statement $F(A) = \emptyset$ where A is a simple abelian variety, whose proof was sketched above. The statement $F(A) = \emptyset$ can be converted into a relative generic vanishing theorem for morphisms with simple kernel A in the sense of [KrW]. Factoring an arbitrary homomorphism $f : X \rightarrow B$ into homomorphisms whose kernels are simple abelian varieties, an iterative application of the assertion $F(A) = \emptyset$ for each of the simple abelian varieties A defining the successive kernels, then easily gives the next

Lemma 1. *Let A be an abelian subvariety of X with quotient map*

$$f : X \rightarrow B = X/A$$

and let $K \in \text{Perv}(X, \mathbb{C})$ be a perverse sheaf on X . Then for a generic character χ the direct image $Rf_(K_\chi)$ is a perverse sheaf on B .*

Corollary 1. *$H^\bullet(X, F_\chi) = 0$ holds for $F \in E(X)$ and generic χ .*

Remark. For finitely many perverse sheaves and a homomorphism $f : X \rightarrow B$, by lemma 1 one always finds characters χ such that $\Gamma^\chi(K) = {}^pH^0(Rf_*(K_\chi))$ is an exact functor on the tensor subcategory \mathcal{T} of $D_c^b(X, \mathbb{C})$ generated under the convolution product (see [KrW]) by these objects, in the sense that Γ^χ maps distinguished triangles to short exact sequences. We use this to analyse stalks: Suppose the stalk $Rf_*(K_\chi)_b$ vanishes for generic χ . Let $F_b = f^{-1}(b)$ be the fiber $F = F_b$. Then $M = (K_\chi)|_{F_b}$ is in ${}^pD^{[-\dim(B), 0]}(F)$ and for generic χ all perverse sheaves $M^i = {}^pH^{-i}(M)$ are acyclic, i.e. $H^\bullet(F, M^i) = 0$. Although M and also the perverse sheaves M^i are not necessarily semisimple, this follows from the exactness of the functors Γ^χ ; here as a consequence of

$$H^i(F, M) = H^0(F, M^i) = H^\bullet(F, M^i)$$

for generic χ identifying F_b with A . The same statement carries over to the irreducible perverse Jordan-Hölder constituents P of the perverse sheaves M^i .

Direct images. For the definition of M we fix a suitable generic character χ chosen as above. Then $M_{\chi_0} = K_{\chi\chi_0}|_{F_b}$ for arbitrary χ_0 gives the ‘cohomology’ spectral sequence

$$\bigoplus_{j+i=k} H^{-j}(F(b), (M^i)_{\chi_0}) \implies R^{-k}f_*(K_{\chi\chi_0})_b.$$

The cohomology sheaves $\mathcal{H}^{i+k}(M^i) = \bigoplus_i \mathcal{H}^k(M^i[i])$ are related to the cohomology sheaves $\mathcal{H}^k(M)$, or equivalently the stalk cohomology sheaves of the complex K_χ at points $x \in F_b$ via the ‘stalk’ spectral sequence with $E_2^{p,q} = \mathcal{H}^p(M^{-q})$

$$\bigoplus_{-p-q=-k} \mathcal{H}^{-p}(M^q) \implies \mathcal{H}^{-k}(M) = \mathcal{H}^{-k}(K_\chi)|_{F_b}$$

on F_b with differentials $d_2 : \mathcal{H}^{i+k-1}(M^{i-1}) \rightarrow \mathcal{H}^{i+k+1}(M^i)$.

$$\begin{array}{cccccccc}
\mathcal{H}^{-d(A)}(M^0) & \dots & \mathcal{H}^{-d-1}(M^0) & \mathcal{H}^{-d}(M^0) & \longrightarrow & \mathcal{H}^{-1}(M^0) & \longrightarrow & \mathcal{H}^0(M^0) & 0 & \dots \\
\mathcal{H}^{-d(A)}(M^1) & \dots & \mathcal{H}^{-d-1}(M^1) & \mathcal{H}^{-d}(M^1) & \dots & \mathcal{H}^{-1}(M^1) & \longrightarrow & \mathcal{H}^0(M^1) & 0 & \dots \\
\dots & & \vdots & & & \vdots & & \vdots & 0 & \dots \\
\dots & & \vdots & & & \vdots & & \vdots & 0 & \dots \\
\dots & & \vdots & & & \vdots & & \vdots & 0 & \dots \\
\mathcal{H}^{-d(A)}(M^d) & \dots & \mathcal{H}^{-d-1}(M^d) & \mathcal{H}^{-d}(M^d) & \dots & \mathcal{H}^{-1}(M^d) & \longrightarrow & \mathcal{H}^0(M^d) & 0 & \dots \\
\dots & & \vdots & \vdots & \vdots & \vdots & & \vdots & 0 & \dots \\
\dots & & \vdots & \vdots & \vdots & \vdots & & \vdots & 0 & \dots \\
\mathcal{H}^{-d(A)}(M^{d(B)}) & \dots & \mathcal{H}^{-d-1}(M^{d(B)}) & \mathcal{H}^{-d}(M^{d(B)}) & \dots & \mathcal{H}^{-1}(M^{d(B)}) & \longrightarrow & \mathcal{H}^0(M^{d(B)}) & 0 & \dots
\end{array}$$

Here $d(A)$ and $d(B)$ are the dimensions of A resp. B . There are edge morphisms $\mathcal{H}^0(M^i) \rightarrow \mathcal{H}^{-i}(M)$ and $\mathcal{H}^{-i}(M) \rightarrow \mathcal{H}^{-i}(M^0)$.

Lemma 2. For simple A and $f : X \rightarrow X/A$ we have $Rf_*(K) = 0 \implies K \notin F(X)$.

Proof. $Rf_*(K) = 0$ implies $Rf_*(K_\chi) = 0$ for generic χ . So all M^i are acyclic on A and hence $\mathcal{H}^{-j}(M^i) = 0$ for $j \neq d(A)$. Thus by the stalk spectral sequence, $\mathcal{H}^{-i-d(A)}(K_\chi)|_{F(b)} \cong \mathcal{H}^{-d(A)}(M^i)$ is a translation invariant sheaf on A and therefore is never a skyscraper sheaf. We apply this for the prime component \mathcal{P}_K of K . According to [W2, lemma 2.1] $Rf_*(K_\chi) = 0$ for generic χ implies $Rf_*(\mathcal{P}_{K\chi}) = 0$ for generic χ and $K \in F(X)$ implies $\mathcal{P}_K \in F(X)$. Finally for $\mathcal{P}_K \in F(X)$ the stalk $\mathcal{H}^{-i}(\mathcal{P}_K)$ is a skyscraper sheaf at least for one $i = v_K$ by [W2, lemma 1, part 7]. A contradiction. \square

2 Restriction in steps

Consider exact sequences of abelian varieties

$$0 \longrightarrow B_1 \longrightarrow B \xrightarrow{h} B_2 \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow A \xrightarrow{p} B_1 \longrightarrow 0$$

and a diagram of quotient homomorphisms where $A = g^{-1}(B_1)$ and $p = g|_A$, where C is the kernel of the projection $g : X \rightarrow B$

$$\begin{array}{ccc}
X & \xrightarrow{f} & B_2 = X/A \\
& \searrow g & \nearrow h \\
& B = X/C &
\end{array}$$

Assumption. For perverse K and given quotient morphism $f : X \rightarrow B_2 = X/A$ suppose for generic χ

$$\boxed{R^{-i}f_*(K_\chi) = 0 \quad , \quad \forall i < d} .$$

These vanishing conditions imply acyclicity for the constituents P of the perverse sheaves M^0, M^1, \dots, M^{d-1} .

For $b_2 \in B_2$ and $b_1 \in B_1$ the fibers $C \cong F_{(b_1, b_2)} = g^{-1}(b_1, b_2) \hookrightarrow f^{-1}(b_2) = F_{b_2} \cong A$, can be identified with A respectively with C up to a translation. This being said, we restrict a generic twist K_χ of K (for some generic character $\chi : \pi_1(X, 0) \rightarrow \mathbb{C}^*$) to the fiber F_{b_2}

$$M = M(b_2) = (K_\chi)|_{F_{b_2}} \in {}^pD^{[-\dim(B_2), 0]}(F_{b_2}) ;$$

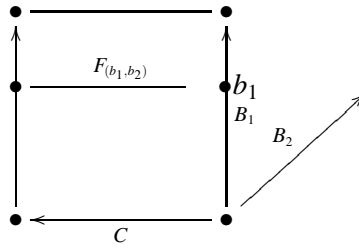
then we further restrict M to $F_{(b_1, b_2)} \hookrightarrow F_{b_2}$ and obtain

$$N = N(b_1, b_2) := M(b_2)|_{F_{(b_1, b_2)}} = K_\chi|_{F_{(b_1, b_2)}} \in {}^pD^{[-\dim(B), 0]}(F_{(b_1, b_2)}) .$$

For $N^k = N^k(b_1, b_2) = {}^pH^{-k}(N)$ in $\text{Perv}(F_{(b_1, b_2)}, \mathbb{C})$ and $M^i = M^i(b_2) = {}^pH^{-i}(M)$ in $\text{Perv}(F_{b_2}, \mathbb{C})$ for $j = 0, \dots, \dim(B_1)$ and $i = 0, \dots, \dim(B_2)$ there is a ‘double restriction’ spectral sequence

$$\bigoplus_{i+j=k} {}^pH^{-j}(M^i(b_2)|_{F_{(b_1, b_2)}}) \implies N^k(b_1, b_2)$$

Picture. The front rectangle visualizes the fiber F_{b_2} , which is isomorphic to $A = \text{Kern}(f)$, and the fiber $F_{(b_1, b_2)} \subset F_{b_2}$ isomorphic to the abelian variety $C = \text{Kern}(g)$.



Now fix $b_2 \in B_2$. Then for almost all closed points $b_1 \in B_1$ the perverse sheaf

$${}^pH^0(M^d(b_2)|_{F_{(b_1, b_2)}})$$

is zero, since it defines a perverse quotient sheaf of $M^d(b_2)$ on F_{b_2} with support in $F_{(b_1, b_2)}$. Indeed these supports are disjoint and there are only finitely many constituents. Furthermore the perverse constituents of the sheaves $M^0(b_2), \dots, M^{d-1}(b_2)$ on $F_{b_2} \cong A$ are in $E(F_{b_2})$, by the vanishing assumption on the direct images: $Rf^{-i}(K_\chi) = 0$ for $i < d$.

For generic χ we have the ‘relative cohomological’ spectral sequence

$$\bigoplus_{j+i=k} H^0(F_{(b_1, b_2)}, {}^p H^{-j}(M^i|_{F_{(b_1, b_2)}})) \implies R^{-k} g_*(K_\chi)_{(b_1, b_2)} .$$

Notice $i = 0, 1, \dots, \dim(B_2)$ and $j = 0, \dots, \dim(B_1)$, where the case $j = 0$ plays a special role as explained above. The spectral sequence is obtained from the double restriction spectral sequence combined with the degenerate cohomology spectral sequence for g using $H^0(F_{(b_1, b_2)}, N^k) = R^{-k} g_*(K_\chi)_{(b_1, b_2)}$ for generic χ . Now assume

$$\boxed{d = \dim(B_1) = \mu(A) = \mu(X)} .$$

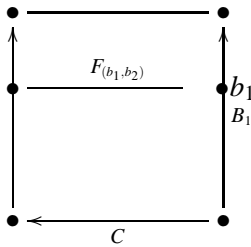
Proposition 1. *For $d = \mu(A) = \dim(B_1) = \mu(X)$ suppose given an irreducible perverse sheaf K with the vanishing condition $R^{-i} f_*(K_\chi)_{b_2} = 0$ for $i < d$ and generic χ . Then for fixed $b_2 \in B_2$ and generic $\chi : \pi_1(X, 0) \rightarrow \mathbb{C}^*$ we have an exact sequence*

$$H^0(F_{(b_1, b_2)}, {}^p H^0(M^d|_{F_{(b_1, b_2)}})) \rightarrow R^{-d} g_*(K_\chi)_{(b_1, b_2)} \rightarrow H^0(F_{(b_1, b_2)}, {}^p H^{-d}(M^0|_{F_{(b_1, b_2)}})) .$$

In particular $R^{-d} g_(K_\chi)_{(b_1, b_2)} = 0$ holds for almost all $b_1 \in B_1$ (for fixed $b_2 \in B_2$), if M^0 vanishes. Notice $M^0 = 0$ iff K does not have support in the fiber F_{b_2} .*

Proof. $\bigoplus_{j+i=d} H^0(F_{(b_1, b_2)}, {}^p H^{-j}(M^i|_{F_{(b_1, b_2)}})) \implies R^{-d} g_*(K_\chi)_{(b_1, b_2)}$ for generic χ and $k = d$ degenerates by Lemma 3 which shows that for $0 < i < d$ we can ignore all terms $j = 1, \dots, d - 1 = \dim(B_1) - 1$ in this spectral sequence. Since $j + i = k$, for $k = d$ only the terms $(j, i) = (0, d)$ and the term $(j, i) = (d, 0)$ remain. This proves our assertion. \square

Before we give the proof of the lemma, recall that the abelian variety A can be identified with the ‘front rectangle’ F_{b_2} , the fiber of b_2 for fixed $b_2 \in B_2$, which contains $F_{(b_1, b_2)}$



The irreducible constituents P of the perverse sheaves $M^i, i < d$ are acyclic perverse sheaves living on the ‘front rectangle’ A and their irreducible perverse constituents P are acyclic.

Lemma 3. *Suppose $\mu(A) = \dim(B_1) = \mu(X) = d$. Then B_1 is simple and for $i = 0, 1, \dots, d-1$ and the constituents P of M^i for generic χ we have*

$$H^\bullet(F_{(b_1, b_2)}, {}^p H^{-j}(P|_{F_{(b_1, b_2)}})_\chi) = 0 \quad , \quad \forall j = 0, 1, \dots, d-1 .$$

Proof. The irreducible constituents $P = \tilde{P}_\chi$ (for $K|_{F_{b_2}}$) of the M^i for $i = 0, 1, \dots, d-1$ are acyclic on A and for $p : A \rightarrow B_1$ the direct image $Rp_*(P)$ is perverse (χ being generic) so that therefore $\chi(Rp_*(P)) = \chi(P) = 0$ holds. Since B_1 is simple of dimension $\dim(B_1) = d$, the semisimple perverse sheaf $Rp_*(P)$ in $E(B)$ is of the form

$$Rp_*(P) = \bigoplus_{\psi \in \Psi(\chi)} m_\psi \cdot \delta_{B_1}^\psi .$$

Then $R^{-i}p_*(P)_{b_1} = 0$ for $i \neq d = \dim(B_1)$ and all $b_1 \in B_1$, so that for generic χ the perverse sheaves ${}^p H^{-j}(P|_{F_{(b_1, b_2)}})$ are acyclic for $j = 0, \dots, -d+1$ by the remark on page 5 applied for (P, p) instead of (K, f) . Thus $H^\bullet(F_{(b_1, b_2)}, {}^p H^{-j}(P|_{F_{(b_1, b_2)}})_\chi)$ vanishes for $j = 0, \dots, d-1$. \square

3 Stalk vanishing conditions

Let $\mu(X)$ be the minimum of the dimensions of the simple abelian variety quotients B of X . We say an abelian quotient variety B of X is *minimal*, if $\dim(B) = \mu(X)$. For a sheaf complex P on X define

$$\mu(P) = \max\{v \mid \mathcal{H}^{-i}(P) = 0 \text{ for all } i < v\} .$$

Lemma 4. $\mu(K) \geq \mu(X)$ holds for $K \in E(X)$.

Proof by induction on $\dim(X)$. Choose $f : X \rightarrow B$ with simple minimal B . Then $Rf_*(K_\chi)$ is perverse for generic χ and hence $Rf_*(K_\chi) \in E(B)$ is of the form $\bigoplus_\psi m_\psi \cdot \delta_B^\psi$. By the induction hypothesis we can assume $M^0 = 0$, since otherwise the support of K is contained in a proper abelian subvariety of X . $M = K_\chi|_{f^{-1}(b)}$ has acyclic perverse cohomology sheaves $M^i \in E(\text{Kern}(f))$ for $i = 1, \dots, d-1$ and $d = \mu(X)$. Then by induction $\mu(M^i) \geq \mu(\text{Kern}(f)) \geq \mu(X) = d$ implies $\mathcal{H}^{-v}(M^i) = 0$ for all $i = 0, \dots, \dim(B) - 1 = d - 1$ and all $v < d = \mu(X)$. Hence $\mu(K) \geq d$ by the stalk spectral sequence discussed in section 1. \square

In the next lemma we give some information about prime components \mathcal{P}_K of perverse sheaves $K \in F(X)$. For details on prime components we refer to [W2].

Lemma 5. *For $K \in F(X)$ the following holds*

1. *The prime component $\mathcal{P}(K)$ is in $F(X)$ with $v_K := \mu(\mathcal{P}_K) \geq \mu(X)$.*
2. *$K \in F_{\max}$ implies $\mathcal{P}_K \in F_{\max}$ and $v_K = \mu(\mathcal{P}_K) = \mu(X)$.*
3. *For $K \in F_{\max}(X)$ and any minimal quotient $B = X/A$ of X the restriction $\mathbf{M} = \mathcal{P}_K|_F$ of \mathcal{P}_K to any fiber $F = f^{-1}(b)$ is a complex with Euler perverse cohomology M^i for $i = 0, \dots, \mu(X) - 1$. For the fiber over the point $b = 0$ furthermore $M^{\mu(X)}$ has a nontrivial perverse skyscraper quotient concentrated at the point zero.*

Proof. For the first assertion $\mathcal{P}(K) \in F(X)$ see [W2, lemma 2.5]. $\mu(\mathcal{P}_K) \geq \mu(X)$ holds by lemma 4 and $v_K := \mu(\mathcal{P}_K)$ and [W2, lemma 1, part 7]. Hence $v_K \geq \mu(X)$. By [W2, lemma 4] on the other hand $v_K \leq \mu(X)$ for $K \in F_{\max}(X)$. Hence $v_K = \mu(\mathcal{P}_K) = \mu(X)$ holds for $K \in F_{\max}(X)$. The assertion on the skyscraper subsheaf comes from the edge term of the above spectral sequence, since for $\mathcal{P}_K \in F(X)$ the cohomology $\mathcal{H}^{-v(K)}(\mathcal{P}_K)$ is a skyscraper sheaf. $K \in F_{\max}(X)$ implies $\mathcal{P}_K \in F_{\max}(X)$, since $Rf_*(K_\chi)$ is perverse for generic χ , and hence $Rf_*(K_\chi) = 0$ iff $Rf_*(\mathcal{P}_{K\chi}) = 0$ by [W2, lemma 2.1]. \square

Lemma 6. *For $K \in F_{\max}(X)$ the support of K is not contained in a translate of a proper abelian subvariety A of X .*

Proof. For the projection $f : X \rightarrow X/A$ the support Z of $Rf_*(\mathcal{P}_K)$ becomes zero: $f(Z) = \{0\}$. This also holds for generic twists of K , so we could assume that $Rf_*(\mathcal{P}_K)$ is perverse. Therefore $Rf_*(\mathcal{P}_K) = 0$, since otherwise for a skyscraper sheaf $\chi(Rf_*(\mathcal{P}_K)) > 0$ would hold, a contradiction. This implies $Rf_*(K_\chi) = 0$ for generic χ contradicting the maximality of K . \square

4 Supports

Let A be an abelian subvariety of X and K be an irreducible perverse sheaf on X . For quotient homomorphisms $f : X \rightarrow B = X/A$ consider the assertions

1. K is C -invariant for some nontrivial abelian subvariety C of A .

2. $Rf_*(K_\chi) = 0$ for generic χ .

3. $Rf_*(K_\chi) = 0$ on a fixed dense open subset W of $f(Z)$ for generic χ .

Obviously $1. \implies 2. \implies 3.$ We remark that $Rf_*(K_{\chi_0}) = 0$ for a single χ_0 implies 2. using corollary 1. Indeed the fibers M are acyclic and remain acyclic for generic twist.

Proposition 2. *Suppose $F(A) = \emptyset$ and suppose $A + Z = Z$ for the support Z of K . Then all three properties 1, 2 and 3 from above are equivalent. For $K \in F(X)$ then furthermore $\text{supp}(Rf_*(K_\chi)) = f(\text{supp}(K))$ holds for generic χ .*

Step 1) Fix a smooth dense open subset $W \subset f(Z) = Z/A$ of dimension d so that $K|_U = E[\dim(U)]$ for a local system E on a dense Zariski open subset U of $f^{-1}(W)$, and so that $U \cap f^{-1}(b)$ is dense in every fiber for $b \in W$; fix a closed point b in W . Then $F = f^{-1}(b)$ can be identified with A . For $M = (K_\chi)|_F \in {}^pD^{[-d,0]}(F)$ and $M^i = {}^pH^{-i}(M)$ for $i = 0, \dots, d$ notice $M^d \neq 0$, since ${}^pH^{-d}(M)$ contains the intermediate extension of $E_{F \cap U}[g]$ as constituent.

Step 2) Let C be a nontrivial abelian subvariety of A and $g : X \rightarrow X/C$ the projection. By assumption K is simple and g is smooth with connected fibers of dimension $\dim(C)$, so by [BBD, p.108ff] K or equivalently K_χ is C -invariant for some nontrivial abelian subvariety C of A and some character χ iff

$${}^pH^{-\dim(C)}(Rg_*(K_\chi)) \neq 0.$$

Step 3) For fiber inclusions $i : F \hookrightarrow Z$ and $i_C : F/C \hookrightarrow Z/C$

$$\begin{array}{ccccc} & & f & & \\ & \nearrow & & \searrow & \\ Z & \xrightarrow{g} & Z/C & \longrightarrow & Z/A \\ \uparrow i & & \uparrow i_C & & \uparrow i_b \\ F & \xrightarrow{g} & F/C & \longrightarrow & \{b\} \end{array}$$

proper base change gives $i_C^* Rg_*(K_\chi) = Rg_*(i^*(K_\chi)) = Rg_*(M)$. In order to compute ${}^pH^{-\dim(C)-d}(Rg_* i^*(-))$ use that under the functor Rg_* perversity drops at most by $-\dim(C)$ (see [BBD, 4.2.4])

$$Rg_* : {}^pD^{\geq n}(-) \rightarrow {}^pD^{\geq n-\dim(C)}(-)$$

and that therefore ${}^pH^{-\dim(C)}Rg_* : \text{Perv}(-, \mathbb{C}) \rightarrow \text{Perv}(-, \mathbb{C})$ is left exact. Also i^* and i_C^* drop perversity at most by $-d$. By proper base change $i_C^* \circ Rg_* = Rg_* \circ i^*$. For perverse K the distinguished truncation triangle $({}^pH^{-\dim(C)}(Rg_*(K))[\dim(C)], Rg_*(K), K')$ gives $K' \in {}^pD^{>-\dim(C)}(Z/C, \mathbb{C})$ and $i_C^*(K') \in {}^pD^{>-\dim(C)-d}(Z/C, \mathbb{C})$. Therefore

$${}^pH^{-d}(i_C^*({}^pH^{-\dim(C)}(Rg_*(K)))) \cong {}^pH^{-\dim(C)-d}(i_C^*Rg_*(K))$$

This gives the upper (and similarly the lower) part of the next commutative diagram

$$\begin{array}{ccc} \text{Perv}(Z, \mathbb{C}) & \xrightarrow{{}^pH^{-\dim(C)} \circ Rg_*} & \text{Perv}(Z/C, \mathbb{C}) \\ \downarrow {}^pH^{-d} \circ i^* & \searrow {}^pH^{-\dim(C)-d} \circ i_C^* \circ Rg_* & \downarrow {}^pH^{-d} \circ i_C^* \\ \text{Perv}(F, \mathbb{C}) & \xrightarrow{{}^pH^{-\dim(C)} \circ Rg_*} & \text{Perv}(F/C, \mathbb{C}) \end{array}$$

so that

$${}^pH^{-\dim(C)}(Rg_*({}^pH^{-d}i^*(K_\chi))) = {}^pH^{-\dim(C)}(Rg_*(M^d)) \neq 0$$

implies ${}^pH^{-d}(i_C^*({}^pH^{-\dim(C)}Rg_*(K_\chi))) \neq 0$, and hence as required for step 2)

$${}^pH^{-\dim(C)}(Rg_*(K_\chi)) \neq 0 .$$

Step 4) For the proof of the proposition for generic characters χ now suppose

$$Rf_*(K_\chi)_b = 0 \quad , \quad \text{at } b \in Z/A .$$

From step 3 in the case $C = A$ therefore M is acyclic on F but not zero, using $M^d \neq 0$. For a suitable χ all perverse cohomology sheaves $M^i = {}^pH^i(M)$ are acyclic on F and their perverse Jordan-Hölder constituents as well. Fix one such χ and notice ${}^pH^{-i}(M_{\chi_0}) = {}^pH^{-i}(M)_{\chi_0} = M_{\chi_0}^i$ for any character χ_0 .

Step 5) Since $F(A) = \emptyset$ holds by assumption, all the acyclic perverse Jordan-Hölder constituents of the perverse sheaves M^i are negligible perverse sheaves on $F \cong A$. Let $S \hookrightarrow M^d \neq 0$ be a simple perverse subobject. Then $S = (\delta_C)_{\chi_0^{-1}} * L$ for some $L \in M(A)$ and some nontrivial abelian subvariety C of A and some character χ_0 . For the corresponding projection $g : X \rightarrow X/C$

$${}^pH^{-\dim(C)}(Rg_*(S_{\chi_0})) \hookrightarrow {}^pH^{-\dim(C)}(Rg_*(M_{\chi_0}^d))$$

by the left exactness of ${}^pH^{-\dim(C)}$. Furthermore ${}^pH^{-\dim(C)}(Rg_*(S_{\chi_0})) \neq 0$. For the last assertion notice $Rg_*(S_{\chi_0}) = H^\bullet(C) \otimes_{\mathbb{C}} Rg_*(L)$ and hence

$${}^pH^{-\dim(C)}(Rg_*(S_{\chi_0})) = H^{-\dim(C)}(C) \otimes_{\mathbb{C}} ({}^pH^0(Rg_*(L))) .$$

Indeed L has nonvanishing Euler characteristic and therefore $Rg_*(L) \neq 0$. Then ${}^pH^i(Rg_*(L)) = 0$ holds for all $i \neq 0$, since ${}^pH^i(Rg_*(S_{\chi_0})) = 0$ for $i < -\dim(C)$. This being said, we obtain

$${}^pH^{-\dim(C)}(Rg_*(M_{\chi_0}^d)) \neq 0 .$$

Hence our assertion follows from step 1)-3) applied for $\chi\chi_0$ instead of χ . \square

Remark. For an abelian variety X let A be the connected stabilizer of an irreducible subvariety Z of X , and let \tilde{Z} be the image of Z in $\tilde{X} = X/A$. Then the connected stabilizer of \tilde{Z} in \tilde{X} is trivial.

Proposition 3. Assume $F(B) = \emptyset$ for all quotients B of X of dimension $< \dim(X)$. Then any $K \in F(X)$ is in $F_{\max}(X)$.

Proof. Suppose $K \in F(X)$ but $K \notin F_{\max}(X)$. Then there exists a minimal quotient $p : X \rightarrow B$ such that $Rp_*(K_\chi) = 0$ holds (χ generic). The fiber perverse sheaves $M^i(b), b \in B$ and their Jordan-Hölder constituents P then are all acyclic perverse sheaves on the abelian variety $\text{Kern}(p)$. Hence by the induction assumption these P are in $N(\text{Kern}(p))$, hence have degenerate support. Since $Z = \text{supp}(K) = \bigcup_{b \in B, i} \text{supp}(M^i(b))$, by [A] therefore Z is degenerate, i.e. there exists an abelian subvariety A of X of dimension > 0 such that $Z + A = Z$. Suppose $A \neq X$. For the quotient morphism $f : X \rightarrow X/A$ and by the induction assumption $F(A) = \emptyset$ then $Rf_*(K_\chi) \neq 0$ holds for generic χ , since otherwise $K \notin F(X)$ by proposition 2. Since $Rf_*(K_\chi)$ is perverse for generic χ , therefore $L = Rf_*(K_\chi) \in E(X/A)$. By proposition 2 furthermore $\text{supp}(L) = \tilde{Z}$ for $\tilde{Z} = f(Z)$; in particular \tilde{Z} is irreducible. Since $L \in E(X/A)$ and since $F(X/A) = \emptyset$ holds by the induction assumption, the support \tilde{Z} of L is a finite union of degenerate subvarieties. Since \tilde{Z} is irreducible, therefore \tilde{Z} is degenerate. This is a contradiction, since by the remark above for the quotient $f : X \rightarrow B = X/A$ by the connected stabilizer A the irreducible variety $\tilde{Z} = f(Z)$ has trivial connected stabilizer. This shows $Z = X$. But then Z is invariant under $\text{Kern}(p)$, and the vanishing $Rp_*(K_\chi) = 0$ for generic χ implies $K \notin F(X)$ by proposition 2 applied for the morphism $p : X \rightarrow B$. \square

Corollary 2. $F(X) = F_{\max}(X)$ for X isogenous to $A_1 \times A_2$ with simple factors A_1, A_2 of dimension $\dim(A_1) = \dim(A_2) = d$.

Remark. In the situation of the last corollary for irreducible $K \in E(X)$ either $v_K = d$ or $K \cong \delta_X^\Psi$. Indeed $K \in F_{\max}$. So $v_K = d$ by lemma 5. For $K \in N(X)$ we know $v_K = d$ or $2d$.

Although we do not need this for the proof, in the remaining part of this section we show that under similar conditions for $K \in F(X)$ the support of K is X . For this we first state a result of [W].

Theorem 3. *For complex abelian varieties the support Z of an irreducible perverse sheaves $K \in E(X)$ always is a degenerate irreducible subvariety of X , i.e. $A + Z = Z$ holds for some abelian subvariety A of X of dimension > 0 .*

Corollary 3. *$K \in E(X) \implies K \in N(X)$ for irreducible $K \in \text{Perv}(X, \mathbb{C})$, provided $F(A) = \emptyset$ holds where A is the connected stabilizer of the support Z of K .*

Proof. By theorem 3 the support Z of K is an irreducible degenerate subvariety of X . For the quotient $f : X \rightarrow B = X/A$ by the connected stabilizer A the irreducible variety $\tilde{Z} = f(Z)$ by construction has trivial connected stabilizer. If for generic χ the perverse sheaf $Rf_*(K_\chi)$ vanishes, for $K \in F(X)$ and $F(A) = \emptyset$ our assertion follows from proposition 2. So it suffices to show $Rf_*(K_\chi) = 0$. Suppose $Rf_*(K_\chi) \neq 0$. Then $\tilde{Z} = \text{supp}(Rf_*(K_\chi))$ by proposition 2. Since $\chi(Rf_*(K_\chi)) = \chi(K) = 0$, any irreducible constituent P of $Rf_*(K_\chi)$ is in $E(X/A)$ and therefore $\text{supp}(P)$ is degenerate by theorem 3. Since $\tilde{Z} = \text{supp}(Rf_*(K_\chi)) = \bigcup \text{supp}(P)$ and \tilde{Z} is irreducible, we obtain $\tilde{Z} = \text{supp}(P)$ for some P . Hence \tilde{Z} is degenerate. A contradiction. \square

An immediate consequence of these arguments is

Proposition 4. *Assume $F(B) = \emptyset$ for all quotients B of X of dimension $< \dim(X)$. Then any $K \in F(X)$ has support X .*

Proof. The assertion $\text{supp}(K) = X$ follows from corollary 3. To show $K \in F_{\max}(X)$ consider $f : X \rightarrow B$ for minimal B . Suppose $Rf_*(K_\chi) = 0$ holds for generic χ . Then proposition 2 can be applied for K and $A = \text{Kern}(f)$, since $F(A) = \emptyset$ and $\text{supp}(K) = X$ is invariant under A . This proves $K \in N(X)$. A contradiction. \square

5 Main Theorem

Lemma 7. *$F_{\max}(X) = \emptyset$, if X has a simple quotient B with $\dim(B) > \mu(X)$.*

Proof. Suppose $K \in F_{\max}(X)$. For $f : X \rightarrow B$ then $\mu(M^i) \geq \mu(X)$ for $i = 0, \dots, \dim(B) - 1$ by lemma 4, since these M^i are acyclic. Therefore $\mathcal{H}^{-\mu(X)} K_\chi|_{f^{-1}(0)} \cong \mathcal{H}^{-\mu(X)}(M) \cong \mathcal{H}^{-\mu(X)}(M^0) = 0$. Indeed, for $K \in F_{\max}(X)$ the support is not contained in a translate of a proper abelian subvariety of X by lemma 6, so $M^0 = 0$. This shows

$$\mu(K) > \mu(X) .$$

Now also $\mathcal{P}_K \in F_{\max}(X)$ by lemma 5. The last inequality applied for \mathcal{P}_K instead of K leads to a contradiction. Indeed $\mu(\mathcal{P}_K) = v_K = \mu(X)$ holds by the maximality of K using lemma 5. \square

At least three simple constituents. Consider quotients $g : X \rightarrow B$ with nontrivial kernel C , where B has two simple factors B_1 and B_2 and

$$0 \rightarrow B_1 \rightarrow B \rightarrow B_2 \rightarrow 0 .$$

By lemma 7 we may assume that all simple factors of X have the same dimension $d = \mu(X)$. Put $A = g^{-1}(B_1)$. Consider $K \in F(X)$. Then for $f : X \rightarrow X/A = B_2$ and generic χ the perverse sheaf $Rg_*(K_\chi)$ is in $E(B)$ and hence of the form

$$Rg_*(K_\chi) = \bigoplus_{i \in I} T_{b_i}^*(\delta_{A_i}^{\psi_i}) * M_i \quad \oplus \quad \text{objects in } F(B) \quad \oplus \quad \text{rest}$$

for finitely many simple abelian subvarieties A_i of $B = B_1 \times B_2$ of dimension $\dim(A_i) = \mu(X) = d$. Here b_i are certain points in B , $M_i \in M(B)$ where ψ_i are certain characters and $\delta_A^\psi := \mathbb{C}_A[\dim(A)]_\psi$. The term ‘rest’ denotes the B -invariant term.

Claim. For $K \in F_{\max}(X)$ the index set I is empty.

Proof. Assume I is not empty. For generic χ the perverse sheaf $Rg_*(\mathcal{P}_{K,\chi})$ admits for each $i \in I$ a nontrivial morphism in the derived category ([W2, prop.1])

$$Rg_*(\mathcal{P}_{K,\chi})[v_K] \rightarrow \mathcal{P}_{P_i}[v_{P_i}]$$

for each summand $P_i = T_{b_i}^*(\delta_{A_i}^{\psi_i}) * M_i$. Since $v_K = \mu(X) = d$ (this holds for maximal K by lemma 5) and also $v_{P_i} = d$ for $i \in I$, these morphisms define nontrivial morphisms of perverse sheaves

$$Rg_*(\mathcal{P}_{K,\chi}) \longrightarrow \mathcal{P}_{P_i} = \delta_{A_i}^{\psi_i} ,$$

which then are epimorphisms. Therefore

$$Rg_*(\mathcal{P}_{K,\chi}) = \bigoplus_{i \in I} \delta_{A_i}^{\psi_i} \quad \oplus \quad \text{others} .$$

Since $P = \mathcal{P}_{K,\chi}$ also satisfies $v_P = v_K = d$ and also is in $F(X)$, we may replace K by \mathcal{P}_K . Thus it suffices to show that no terms $\delta_{A_i}^{\psi_i}$ can appear in $Rg_*(K_\chi)$ under our assumptions on K above. For this we use proposition 1, which implies

$$R^{-d}g_*(K_\chi)_{b'_1, b'_2} = 0$$

for almost all $b'_1 \in B'_1$ (for fixed $b'_2 \in B'_2$). Indeed we have to choose the map f used in this proposition 1, so that B_1 from that proposition is $B'_1 = g^{-1}(A_i)$ for some fixed i (and not our fixed B_1). Then for $b'_2 = 0$ the fiber

$$R^{-d}g_*(K_\chi)_{b'_1, 0} = \mathcal{H}^{-d}(\delta_{A_i}^{\psi_i})_{b'_1, 0}$$

is not zero for almost all $b'_1 \in A_i$ contradicting proposition 1, if $I \neq \emptyset$. This proves the claim. Indeed the support and vanishing assumptions from page 7 are satisfied for \mathcal{P}_K with $d = \mu(X)$. Recall that the support \mathcal{P}_K must not lie in a proper abelian subvariety for proposition 1. But $K \in F_{\max}(X)$ implies $\mathcal{P}_K \in F_{\max}(X)$ by lemma 5; hence lemma 6 takes care of this. \square

From the preceding dicussion we conclude

Lemma 8. *Suppose $g : X \rightarrow X/A = B$ with B and $A \neq 0$ as above. Then for $K \in F_{\max}(X)$ and generic χ*

$$Rg_*(K_\chi) = \text{objects in } F(B) \oplus \bigoplus_{\psi \in \Psi(\chi)} m_\psi \cdot \delta_B^\psi.$$

Theorem 4. $F(X) = \emptyset$ for complex abelian varieties X .

Proof. We show $F(X) = \emptyset$ by induction on the number of simple factors (or the dimension) of X . For the simple case see [KrW]. The cases with two simple factors will be considered in proposition 6. So assume that X has at least three simple factors and that $F(D) = \emptyset$ already holds for all proper subvarieties or proper quotients D of X . Hence $K \in F_{\max}(X)$ by proposition 3. Then by lemma 7 all simple factors have dimension $\mu(X) = d$, so there exists a quotient $g : X \rightarrow B$ with kernel $A \neq 0$

$$0 \rightarrow B_1 \rightarrow B \rightarrow B_2 \rightarrow 0$$

with simple factors B_1 and B_2 of dimension $d = \mu(X)$. Now $F(B) = \emptyset$ holds by the induction assumption and K is maximal. Hence $Rg_*(K_\chi) = \bigoplus_{\psi \in \Psi(\chi)} m_\psi \cdot \delta_B^\psi$ is B -invariant by lemma 8 for generic χ . But this contradicts the maximality of K , since then $Rf_*(K_\chi) = 0$ vanishes for generic χ . \square

6 Functoriality

Lemma 9. *For simple $A = \text{Kern}(p : X \rightarrow B)$ and $K \in F(X)$ the perverse cohomology sheaves ${}^p H^i(Rp_*(K))$ are in $E(B)$ for all i .*

Proof. Using isogenies one reduces this to the case $X = A \times B$ where p is the projection onto the second factor. Then either $Rp_*(K) = 0$ and there is nothing to prove, or for certain irreducible perverse sheaves P_i and certain integers v_i

$$Rp_*(K) = \bigoplus_i P_i[v_i] \quad , \quad |v_i| < \dim(A) .$$

Suppose for one of the irreducible summands $P_i \notin E(B)$. For the projection q onto the first factor A

$$\begin{array}{ccc} X = A \times B & \xrightarrow{p} & B \\ \downarrow q \quad \chi & & \downarrow \chi \\ A & \longrightarrow & \text{Spec}(\mathbb{C}) \end{array}$$

by lemma 1 there exists a (generic) character χ of $\pi_1(X, 0)$ so that both 1) $Rq_*(K_\chi)$ is perverse and 2) $H^\bullet(B, (P_i)_\chi) = H^0(B, (P_i)_\chi)$ holds for the finitely many P_i . Since $K \in F(X)$, the Euler characteristic of $Rq_*(K_\chi)$ vanishes. Since $Rq_*(K_\chi)$ is perverse, therefore $Rq_*(K_\chi) \in E(A)$ is either zero or of the form $Rq_*(K_\chi) = \bigoplus_{\psi \in \Psi} m_\psi \cdot \delta_A^\psi$ and $H^\bullet(A, Rq_*(K_\chi)) = \bigoplus_i H^\bullet(B, (P_i[v_i])_\chi)$. For $P_i \notin E(B)$ the cohomology of all twists $(P_i)_\chi$ does not vanish (since the Euler characteristic is constant > 0 and independent of twists). So for generic χ then $H^\bullet(B, (P_i[v_i])_\chi) = H^{v_i}(B, (P_i)_\chi) \neq 0$. Hence by comparison $Rq_*(K_\chi)$ can not vanish and therefore is a sum of δ_A^ψ so that at least for one character ψ the cohomology $H^\bullet(A, \delta_A^\psi)$ does not vanish, i.e. ψ is trivial. This gives a contradiction, since then $H^\bullet(A, \delta_A^\psi)$ contains terms of degree $\dim(A)$. For all summands P_j , that are in $E(B)$, the cohomology $H^\bullet(B, (P_j[v_j])_\chi)$ vanishes for generic χ by corollary 1. For the others the cohomology $H^{v_i}(B, (P_i)_\chi) \neq 0$ does not contribute to degree $\dim(A)$, since these v_i satisfy $|v_i| < \dim(A)$. \square

7 Convolution

For simple abelian varieties A_1 and A_2 of dimension d and semisimple perverse sheaves K, L on the cartesian product $X = A_1 \times A_2$ consider the diagram

$$\begin{array}{ccccc}
 A_1 \times A_1 & \xleftarrow{p_1 \times p_1} & X \times X & \xrightarrow{p_2 \times p_2} & A_2 \times A_2 \\
 \downarrow a & & \downarrow b & & \parallel \\
 A_1 & \xleftarrow{p_1} & A_1 \times A_2^2 & \xrightarrow{p_{23}} & A_2 \times A_2 \\
 \parallel & & \downarrow c & & \downarrow a \\
 A_1 & \xleftarrow{p_1} & A_1 \times A_2 & \xrightarrow{p_2} & A_2
 \end{array}$$

with the morphisms $a(x, y) = x + y$, $b(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_2, x_4)$ and $c(y_1, y_2, y_3) = (y_1, y_2 + y_3)$ and the projections $p_{23}(y_1, y_2, y_3) = (y_2, y_3)$ resp. $(p_1 \times p_1)(x_1, x_2, x_3, x_4) = (x_1, x_3)$ and $(p_2 \times p_2)(x_1, x_2, x_3, x_4) = (x_2, x_4)$ and $Y := A_1 \times A_2^2$. Then

$$K \otimes L := Rb_*(K \boxtimes L) \quad , \quad Rc_*(K \otimes L) = K * L .$$

By the decomposition theorem $K \otimes L$ is a semisimple complex. By the relative Künneth formula $R(p_i \times p_i)_*(K \boxtimes L) = Rp_{i*}K \boxtimes Rp_{i*}(L)$ and hence

$$Rp_{23*}(K \otimes L) = Rp_{2*}(K) \boxtimes Rp_{2*}(L) .$$

By twisting both perverse sheaves K, L with the same character $\chi = (\chi_1, \chi_2)$ of $\pi_1(X, 0) = \pi_1(A_1, 0) \times \pi_1(A_2, 0)$ the direct images $Rp_{2*}(K)$ and $Rp_{2*}(L)$, and also $Rp_{23*}({}^pH^i(K \otimes L))$ become perverse sheaves on A_2^2 for generic χ_1 (lemma 1). Now $\bigoplus_{i+j=k} {}^pH^j(Rp_{23*}({}^pH^i(K \otimes L))) = {}^pH^k(Rp_{23*}(K \otimes L))$ by the decomposition theorem. The right side vanishes for $k \neq 0$, since $Rp_{23*}(K \otimes L) = Rp_{2*}K \boxtimes Rp_{2*}(L)$ is perverse. So all terms for $k \neq 0$ on the left are zero, the terms $j \neq 0$ vanish for a suitable twist χ_1 . Hence $Rp_{23*}({}^pH^0(K \otimes L)) = Rp_{23*}(K \otimes L) = Rp_{2*}(K) \boxtimes Rp_{2*}(L)$ and $Rp_{23*}({}^pH^i(K \otimes L)) = 0$ for $i \neq 0$ and generic χ_1 . Notation: $K \circ L = {}^pH^0(K \otimes L)$ and $K \bullet L = \bigoplus_{i \neq 0} {}^pH^i(K \otimes L)$. With the decomposition $K \otimes L = (K \circ L) \oplus (K \bullet L)$ we obtain

$$Rp_{23*}(K \otimes L) = Rp_{2*}(K) \boxtimes Rp_{2*}(L) \quad , \quad Rp_{23*}(K \bullet L) = 0 .$$

Notice $Rp_{23*}(P) = 0$ for simple constituents P of the semisimple perverse sheaf $K \bullet L$, hence $P \in E(Y)$. Since $\text{Kern}(p_{23}) = A_1$ is simple and $Rp_{23*}(P) = 0$, lemma 2

implies $P \in N(Y)$. Then there exists an abelian subvariety B and a character ψ and some $M \in M(Y)$ such that $P = \delta_B^\psi * M$ holds, and $Rp_{23*}(P) = Rp_{23*}(\delta_B^\psi) * Rp_{23*}(M) = 0$ implies $Rp_{23*}(\delta_B^\psi) = 0$ so that $B \subset \text{Kern}(p_{23})$. Hence up to a character twist $P = p_{23}^*[d](Q)$ for some $Q \in \text{Perv}(A_2^2, \mathbb{C})$. Indeed $\text{Stab}(P)^0$ is an abelian subvariety of Y , and therefore $Rp_{23*}(P_\chi) = 0$ for generic χ implies $\text{Kern}(p_{23}) \subseteq \text{Stab}(P)^0$. Hence

Lemma 10. *Up to character twists the irreducible constituents of $K \bullet L$ are in $p_{23}^*[d](D_c^b(A_2^2, \mathbb{C}))$.*

Corollary 4. *Up to character twists the irreducible constituents of $Rc_*(K \bullet L)$ are in $p_2^*[d](D_c^b(A_2, \mathbb{C}))$.*

Corollary 5. *For $K, L \in \text{Perv}(X, \mathbb{C})$ up to character twists the irreducible constituents of ${}^p\tau^{>d}(K * L)$ are in $p_2^*[d](D_c^b(A_2, \mathbb{C}))$.*

Proof. For the proof we may twist both K and L by an arbitrary character $\chi = (\chi_1, \chi_2)$ of $\pi_1(X, 0)$, since convolution and therefore also b and c commute with character twists. Hence the claim follows from $Rc_*(K \circ L) \subseteq {}^pD^{[-d, d]}(X)$ and corollary 4. \square

The role of the indices 1 and 2 of the decomposition $X = A_1 \times A_2$ is arbitrary, so by a switch

Corollary 6. *For $K, L \in \text{Perv}(X, \mathbb{C})$ and $|i| > d$ we have ${}^pH^i(K * L) = \bigoplus_{\varphi} m_{i\varphi} \cdot \delta_X^\varphi$.*

Corollary 7. *For $K, L \in F(X)$ all the summands $P[d] \hookrightarrow K * L$ for which P is not X -invariant appear in the form $H^\bullet(\delta_{A_2}) \otimes_{\mathbb{C}} P \hookrightarrow K * L$ so that for some $P \in \text{Perv}(X, \mathbb{C})$ and $Q \in {}^pD^{[-d+1, d-1]}(X)$ the following holds*

$$K * L = \left(H^*(\delta_A) \cdot P \right) \oplus \bigoplus_{i, \varphi \in \Phi(\chi)} m_{i\varphi} \cdot \delta_X^\varphi[-i] \oplus Q.$$

Proof. First assume only $K, L \in \text{Perv}(X, \mathbb{C})$. Then any term $P[d]$ in $K * L$ with perverse P , not invariant under $\text{Kern}(p_2)$, is in $Rc_*(K \circ L)$; since $K \circ L \in \text{Perv}(A_1 \times A_2^2, \mathbb{C})$ and since c is smooth of relative dimension d , hence $c^*(P)[d]$ must be a summand of $K \circ L$ and then $Rc_*(c^*(P)[d]) = H^\bullet(\delta_{A_2}) \cdot P$ is a summand of $Rc_*(K \circ L)$. The same applies for terms $P[d]$ in $Rc_*(K \circ L)$ that are $\text{Kern}(p_2)$ -invariant. But now there might be $\text{Kern}(p_2)$ -invariant terms $P[d]$ not coming from $Rc_*(K \circ L)$ but only from $Rc_*(K \bullet L)$. To exclude this possibility we assume $K, L \in F(X)$ so that we can apply the next corollary to show that these critical summands are X -invariant, hence contained in $\bigoplus_{i, \varphi \in \Phi(\chi)} m_{i\varphi} \cdot \delta_X^\varphi[-i]$. \square

We now allow arbitrary characters χ_1 . Notice $Rb_*(K_{\chi_1} \boxtimes L_{\chi_1}) = Rb_*(K \boxtimes L)_{\chi_1}$ or $(K \otimes L)_{\chi_1} = (K_{\chi_1} \otimes L_{\chi_1})$. For irreducible $P = p_{23}^*[d](Q)$ we can determine Q as a direct summand of $Rp_{23*}(P_{\chi_1})$ for a suitable choice of χ_1 . Now $Rp_{23*}(P_{\chi_1}) = Rp_{2*}(K_{\chi_1}) \boxtimes Rp_{2*}(L_{\chi_1})$. For $K, L \in F(X)$ and arbitrary χ_1 we then get $Rp_{2*}(K_{\chi_1}) = \bigoplus_{i,\psi} m_{\psi i} \cdot \delta_{A_2}^\psi[-i]$ from lemma 9, since A_1 is simple. Then Q has to be A_2^2 -invariant for $K, L \in F(X)$. This proves

Corollary 8. *For $K, L \in F(X)$ the complex $K \bullet L$ is Y -invariant.*

Similarly $Rp_{1*}(K)$ and $Rp_{1*}(L)$ are perverse, by twisting with a generic χ_2 and $\bigoplus_{i+j=k} {}^pH^j(Rp_{1*}({}^pH^i(K \otimes L))) = {}^pH^k(Rp_{1*}(K) * Rp_{1*}(L))$. Indeed from now on we make the

Assumption. $K, L \in F(X)$.

Then as required: 1) For generic χ we have $Rp_{2*}(K)$ and $Rp_{2*}(L)$ are perverse and translation invariant on A_2 ; and similarly $Rp_{1*}(K)$ and $Rp_{1*}(L)$ are perverse and translation invariant on A_1 . 2) Furthermore ${}^pH^i(K \otimes L) = 0$ for $|i| \geq d$. Of course only the case $i = \pm d$ is relevant. If ${}^pH^{\pm d}(K \otimes L) \neq 0$, then $K \otimes L$ is $\text{Kern}(b)$ invariant and hence $K \boxtimes L$ does not lie in $F(X^2)$ contradicting that $K, L \in F(X)$ implies $K \boxtimes L \in F(X^2)$.

By lemma 9 for generic χ_2 (we do not write this twist !)

$$Rp_{1*}(K) = \bigoplus_{\psi} m_{\psi}^K \cdot \delta_{A_1}^{\psi} \quad , \quad Rp_{1*}(L) = \bigoplus_{\psi} m_{\psi}^L \cdot \delta_{A_1}^{\psi}$$

and hence

$$Rp_{1*}(K \otimes L) = H^{\bullet}(\delta_A) \cdot \bigoplus_{\psi} m_{\psi}^K m_{\psi}^L \cdot \delta_{A_1}^{\psi} .$$

Since $Rp_{1*}(K \otimes L) = Rp_{1*} \circ Rc_*(K \otimes L)$, we now compare this with $Rp_{1*}(K * L) = Rp_{1*} \circ Rc_*(K \otimes L)$ using the formula for $K * L$ obtained in corollary 7

$$\begin{aligned} Rp_{1*}(K * L) &= Rp_{1*} \left((H^*(\delta_A) \cdot P) \oplus \bigoplus_{i,\varphi \in \Phi(\chi)} m_{i\varphi} \cdot \delta_X^{\varphi}[-i] \oplus Q \right) \\ &= H^{\bullet}(\delta_A) \cdot \left(Rp_{1*}(P) \oplus \bigoplus_{\varphi'} m_{0\varphi'} \cdot \delta_{A_1}^{\varphi'} \right) \oplus Rp_{1*}(Q) . \end{aligned}$$

The terms $Rp_{1*}(\delta_X^{\varphi}[-i])$ must vanish for $i \neq 0$, since they have no counter part in the comparison! But for $i = 0$ there might be contributions $Rp_{1*}(\delta_X^{\varphi}[-i]) = \delta_{A_1}^{\varphi'}$ for certain φ . Notice, for generic χ_2 the functor Rp_{1*} preserves perversity. Therefore,

making a comparison of the terms of degree d in $H^\bullet(\delta_A)$ (for $A = A_1$ or A_2) first, we immediately get $Rp_{1*}(Q) = 0$ for generic χ_2 . Hence for the constituents of Q are up to twists in $p_1^*[d](\text{Perv}(A_1, \mathbb{C}))$. For those in $N(X)$ the assertion is obvious. For those in $F(X)$ use corollary 2, $F(A_2) = \emptyset$ and proposition 2. Since the decomposition $X = A_1 \times A_2$ is arbitrary, again by switching the indices we obtain the next

Proposition 5. *For $K, L \in F(X)$ there exists $P \in \text{Perv}(X, \mathbb{C})$ and some X -invariant semisimple complex $T \in D_c^b(X, \mathbb{C})$ so that*

$$\boxed{K * L = H^*(\delta_{A_2}) \cdot P \oplus T} \quad \boxed{H^\bullet(\delta_{A_2}) \cdot P \hookrightarrow Rc_*(K \circ L)},$$

and $H^\bullet(\delta_{A_2}) \cdot P = Rc_*(c^*(P)[d]) \subseteq Rc_*(K \circ L)$. If K, L are primes, then we have $P = K$ for $K \cong L$ respectively $P = 0$ for $K \not\cong L$.

Proof. From our discussion it is clear that $P = P_1 \oplus P_2$ decomposes such that $P_1 \in Rc_*(K \circ L)$ and P_2 by corollary 8 is X -invariant. This allows to replace P by P_1 and T by $T \oplus H^\bullet(\delta_{A_2}) \cdot P_2$. The assertion on the precise form of P for primes K and L then follows from [W2, lemma 7] applied to the class $F(X)$ after localization with respect to the hereditary class of X -invariant complexes. \square

8 Sheaves on $X = A \times B$

For a prime perverse sheaf $K \in F(X)$ on $X = A \times B$, for simple abelian varieties A, B with dimension $\dim(A) = \dim(B)$, in the last section we have shown that $K \cong K^\vee$ and $K * K \cong H^\bullet(\delta_B) \cdot K \oplus T$ holds for some $T = \bigoplus_\varphi m_{i\varphi} \delta_X^\varphi[-i]$. Let $p : X \rightarrow B$ denote the projection onto the second factor.

Replacing K by a twist K_χ we can suppose that $m_{i\varphi} = 0$ holds for the trivial character φ . We can furthermore assume that there exists an abelian subvariety $A \hookrightarrow X$ with quotient $p : X \rightarrow B = X/A$ such that all the finitely many φ with $m_{i\varphi} \neq 0$ have nontrivial restriction on A . Assuming this, then $Rp_*(T) = 0$ and therefore $Rp_*(K)^2 = H^\bullet(\delta_B) \cdot Rp_*(K)$. Then $L := Rp_*(K) = \bigoplus m_{i\psi}^K \delta_B^\psi[i]$ by lemma 9. Hence $Rp_*(K)$ must be perverse ($m_{i\psi} = 0$ for $i \neq 0$) so that $\bigoplus_\psi H^\bullet(\delta_B) \cdot m_\psi^2 \cdot \delta_B^\psi \cong \bigoplus_\psi H^\bullet(\delta_B) \cdot m_\psi \cdot \delta_B^\psi$. Both statements are an easy consequence of $L * L \cong H^\bullet(\delta_B) \cdot L$. Therefore

$$Rp_*(K) = \bigoplus_\psi m_\psi \cdot \delta_B^\psi \quad , \quad m_\psi \in \{0, 1\} \quad .$$

Notice, that by replacing K by some other K_χ with the same properties the coefficients m_ψ may of course change. But the generic rank r

$$r = \sum_{\psi} m_{\psi} = \text{rank}(R^{-d}p_*(K))_b = (-1)^{\dim(A)} \chi(H^\bullet(A, M(b)))$$

is independent from the specific character twist and the point $b \in B$, since

$$\chi(H^\bullet(A, M(b))) = \chi(H^\bullet(A, M(b)_{\chi_0}))$$

holds for every (!) character χ_0 . Recall that $H^\bullet(A, M(b)_{\chi_0}) = Rp_*(K_{\chi_0})_b$ holds for every b and every χ_0 .

K and any of its twists K_χ have the same stabilizer. Under our assumptions the stabilizer H of K is a finite subgroup of X , say of order n . Then, for the isogeny $\pi = n \cdot_X : X \rightarrow X$, we may replace K by one of the irreducible components P of $\pi_*(K) = \bigoplus_{\chi \in H^*} P_\chi$, which is a direct sum of $\#H$ simple prime perverse sheaves P_χ with $v_P = v_K$ and each $P \in F(X)$ has trivial stabilizer [W2, cor. 4]. So in the following we may always assume $\mathcal{H}^{-d}(K) = \delta_0$, i.e. that $T_x^*(K) \cong K$ implies $x = 0$.

Theorem 5. *For simple complex abelian varieties A and B of dimension $\dim(A) = \dim(B)$ let denote $X = A \times B$ and let $p : X \rightarrow B$ be the projection on the second factor. For a fixed translation invariant complex T on B consider the set \mathcal{B} of isomorphism classes of perverse sheaves K on X such that*

1. K is an irreducible perverse sheaf in $\text{Perv}(X, \mathbb{C})$.
2. $K^\vee \cong K$.
3. $K * K \cong H^\bullet(\delta_B) \cdot K \oplus T$.
4. The support $\text{supp}(K)$ is not contained in a simple abelian subvariety.
5. $\text{Stab}(K)$ is trivial, i.e. $T_x^*(K) \cong K$ implies $x = 0$.
6. $Rp_*(K)$ is perverse of generic rank $r > 1$.

Then \mathcal{B} is empty.

Proof. For simplicity of exposition we may assume $\text{supp}(K) = X$ (by prop.4) although this is not essential for the argument. We prove this theorem by reducing it to a corresponding statement for base fields of positive characteristic p , using

the method of Drinfeld [D] and [BK], [G]. The conditions defining \mathcal{B} are constructible conditions in the sense of [D, lemma 2.5] and [D, section 3]. If \mathcal{B} were not empty, the argument of [D] therefore provides us with some other $K \in \mathcal{B}$, which now is a \mathfrak{o} -adic perverse sheaf on X for some finite extension ring \mathfrak{o} of an l -adic ring \mathbb{Z}_l with prime element π generating the maximal ideal of \mathfrak{o} , such that furthermore we find a subring $R \subset \mathbb{C}$ finitely generated over \mathbb{Z} so that X and the complex $K \otimes^L \mathfrak{o}/\pi\mathfrak{o}$ is defined over $\text{Spec}(R)$, i.e. the pair $(X, K \otimes^L \mathfrak{o}/\pi\mathfrak{o})$ descends to some $(X_R, K_R^{(\pi)})$, with the following properties: a) The structure morphism $X_R \rightarrow S = \text{Spec}(R)$ is universally locally acyclic with respect to $K_R^{(\pi)}$, b) For every maximal ideal of R with the finite residue field κ and the corresponding strict Henselization V for a geometric point over this maximal ideal and embeddings $R \subset V \subset \mathbb{C}$, such that the conditions of [D, section 4.9]¹ are satisfied for a suitable M_i attached to $K_R^{(\pi)}$ and its convolution square, one has equivalences of categories

$$D_{\{M_i\}}^b(X, \mathfrak{o}) \sim D_{\{M_i\}}^b(X_R \otimes_R V, \mathfrak{o}) \sim D_{\{M_i\}}^b(\bar{X}, \mathfrak{o})$$

so that the structure morphism $f: X_V \rightarrow \text{Spec}(V)$ is universally locally acyclic with respect to $K_V^{(\pi)}$, and c) The reduction \bar{K} of K is an irreducible perverse Weil sheaf on the special fiber \bar{X} with structure morphism $\bar{f}: \bar{X} \rightarrow \text{Spec}(\bar{\kappa})$ defined over some finite extension of κ , and finally d) there are similar equivalences of categories as in b)

$$D_{\{N_i\}}^b(B, \mathfrak{o}) \sim D_{\{N_i\}}^b(B_R \otimes_A V, \mathfrak{o}) \sim D_{\{N_i\}}^b(\bar{B}, \mathfrak{o})$$

for suitable N_i on B_R attached to the perverse cohomology sheaves of the sheaf complex $Rp_*(K_R^{(\pi)})$ and its convolution square similar to [D, 6.2.3] with a commutative diagram $(*)$ similar as in [D, (6.1)]

$$\begin{array}{ccc} D_{\{M_i\}}(X, \mathfrak{o}) & \xrightarrow{\sim} & D_{\{M_i\}}(\bar{X}, \mathfrak{o}) \\ \downarrow Rp_* & & \downarrow R\bar{p}_* \\ D_{\{N_j\}}(B, \mathfrak{o}) & \xrightarrow{\sim} & D_{\{N_j\}}(\bar{B}, \mathfrak{o}) \end{array}$$

Here $D_{\{M_i\}}(X, \mathfrak{o})$ as a full subcategory of $D(X, \mathfrak{o})$ is defined in [D, 4.9] as the inverse 2-limit of subcategories $D_{\{M_i\}}(X, \mathfrak{o}/\pi^r\mathfrak{o}) \subset D_{prf}(X, \mathfrak{o}/\pi^r\mathfrak{o})$ so that $C \in D_{\{M_i\}}(X, \mathfrak{o})$ iff $C \otimes_{\mathfrak{o}}^L \mathfrak{o}/\pi\mathfrak{o}$ is in the thick triangulated subcategory $D^{\{M_i\}}(X, \mathfrak{o}/\pi\mathfrak{o})$ of $D(X, \mathfrak{o}/\pi\mathfrak{o})$ generated by the M_i in a finite set $\{M_i\}$ of fixed complexes $M_i \in D_c^b(X_R, \mathfrak{o}/\pi\mathfrak{o})$. Similarly for \bar{X} respectively for finitely many N_j on B (or \bar{B}). We briefly remark

¹similar to [BBD, lemma 6.1.9] where V instead is chosen non canonically as some strict Henselian valuation ring with center in the maximal ideal.

that in loc. cit. these equivalences of derived categories above over \mathbb{C}, V, \bar{K} are first proved on the level of $\mathfrak{o}/\pi^r \mathfrak{o}$ -coefficients. There one implicitly uses that for perfect complexes K of $\mathfrak{o}/\pi^r \mathfrak{o}$ -sheaves (or projective limits of perfect complexes of such complexes i.e. in the sense of [KW, p.96f]) the distinguished triangles $(K \otimes_{\mathfrak{o}} \mathfrak{o}/\pi^{r-s} \mathfrak{o}, K \otimes_{\mathfrak{o}} \mathfrak{o}/\pi^r \mathfrak{o}, K \otimes_{\mathfrak{o}} \mathfrak{o}/\pi^s \mathfrak{o})$ for $0 \leq s \leq r$ show that $K \otimes_{\mathfrak{o}} \mathfrak{o}/\pi^r \mathfrak{o}$ is contained in $D^{\{M_i\}}(X, \mathfrak{o}/\pi^r \mathfrak{o})$, if $K \otimes_{\mathfrak{o}} \mathfrak{o}/\pi \mathfrak{o}$ is contained in $\{M_i\}$ or $D^{\{M_i\}}(X, \mathfrak{o}/\pi \mathfrak{o})$.

The first condition a) can be achieved by a suitable localization of R using [Fin, thm. 2.13]. The acyclicity conditions in loc. cit. $i^* \mathcal{F} \cong i^* R \bar{j}_* \mathcal{F}$ are formulated for constructible $\mathfrak{o}/\pi^r \mathfrak{o}$ -sheaves only, but using truncation with respect to the standard t -structure they extend to $\mathfrak{o}/\pi^r \mathfrak{o}$ -adic complexes K with bounded constructible cohomology sheaves. Since L_{χ} is smooth on X_R for any N -torsion character χ , then

$$f : X_R \rightarrow S = \text{Spec}(R)$$

is also universally locally acyclic with respect to $K_R^{(\pi)} \otimes_{\mathfrak{o}} L_{\chi}$. Here χ is viewed as a character $\chi : \pi_1(X, 0) \rightarrow \mathfrak{o}(\zeta_N)^* \subset \text{Gl}_{\mathfrak{o}}(\mathfrak{o}[\zeta_N])$. Now, if two of three complexes in a distinguished triangle are universally locally acyclic, then also the third is. This remark implies that all $K_V \otimes_{\mathfrak{o}} L_{\chi} \otimes_{\mathfrak{o}}^L \mathfrak{o}/\pi^r \mathfrak{o}$ are universally locally acyclic for $r = 1, 2, \dots$ and our given \mathfrak{o} -adic perverse sheaf $K \in \mathcal{B}$, if it is represented by the system $(K_r)_{r \geq 1}$ of perfect complexes K_r such that $K_r \cong K \otimes_{\mathfrak{o}}^L \mathfrak{o}/\pi^r \mathfrak{o}$.

Now we may also consider K as an object in the category $D_c^b(X, Q)$ or in the category $D_c^b(\bar{X}, Q)$ for $Q = \text{Quot}(\mathfrak{o})$. No matter in which way $K \in \mathcal{B} \implies \chi(K) = 0$ [notice $\chi(K)^2 = \chi(H^{\bullet}(\delta_B)) \cdot \chi(K) + \chi(T) = 0$, since $\chi(T) = 0$ and $\chi(H^{\bullet}(\delta_B)) = 0$.] Furthermore $K \in \mathcal{B}$ implies $K \cong \mathcal{P}_K$ and $v_K = \dim(A)$. [Indeed $K^{\vee} * K \cong K * K \cong H^{\bullet}(\delta_B) \cdot K \oplus T$ implies that \mathcal{P}_K is either a summand of T or as an indecomposable constituent of $H^{\bullet}(\delta_B) \cdot K$ isomorphic to K . In the first case K is translation invariant under X , which is impossible by property 5). Therefore $K \cong \mathcal{P}_K$. Since $\mathcal{P}_K[\pm v_K] \hookrightarrow H^{\bullet}(\delta_B) \cdot K$, this implies $v_K \leq \dim(A)$. On the other hand $\mu(K^{\vee} * K) = 0$ and $K[-\dim(A)] \hookrightarrow K^{\vee} * K$ imply $v_K = \mu(K) \geq \dim(A)$. Hence $v_K = \dim(A)$.]

For characters $\chi = (\chi_1, \chi_2)$ of $\pi_1(X, 0) = \pi_1(A, 0) \times \pi_1(B, 0)$ notice that over the base field \mathbb{C} we know that for almost all characters χ_1 of $\pi_1(A, 0)$ the direct image $Rp_*(K_{\chi_1}) \in D_c^b(B, Q)$ must be a locally free translation invariant perverse sheaf on B of rank r . For fixed χ_1 therefore $\mathcal{H}^{\bullet}(Rf_*(K_{\chi})) = H^{\bullet}(B, Rp_*(K_{\chi_1})_{\chi_2})$, considered with coefficients in \mathfrak{o} , is a \mathfrak{o} -torsion module for almost all characters χ_2 of $\pi_1(B, 0)$. Now for $K \in D_c^b(X, \mathfrak{o})$ this information is entirely encoded in the exact sequences for $r \rightarrow \infty$

$$0 \rightarrow \mathcal{H}^n(Rf_*(K_{\chi})) / \pi^r \rightarrow \mathcal{H}^n(Rf_*(K_{\chi} \otimes_{\mathfrak{o}}^L \mathfrak{o}/\pi^r \mathfrak{o})) \rightarrow \mathcal{H}^{n+1}(Rf_*(K_{\chi}))[\pi^r] \rightarrow 0$$

in the form that $\mathcal{H}^\bullet(Rf_*(K_\chi \otimes^\mathbb{L} \mathfrak{o}/\pi^r \mathfrak{o}))$ has bounded length independent from r . Now consider the reduction (\bar{X}, \bar{K}) of (X, K) , defined over the algebraic closure $\bar{\kappa}$ of the finite residue field κ of R with respect to the maximal ideal of V , and the base change ring homomorphisms $V \hookrightarrow \mathbb{C}$ and $V \rightarrow \bar{\kappa}$. By the universal local acyclicity of the structure morphism f for all $K_V \otimes_{\mathfrak{o}} L_\chi \otimes_{\mathfrak{o}}^\mathbb{L} \mathfrak{o}/\pi^r \mathfrak{o}$ the above bounded length conditions over R or V are inherited to the reduction, so that $\mathcal{H}^\bullet(R\bar{f}_*(\bar{K}_\chi \otimes^\mathbb{L} \mathfrak{o}/\pi^r \mathfrak{o}))$ has bounded length independent from r , again for almost all characters χ_2 of $\pi_1(\bar{B}, 0) \cong \pi_1(B, 0)$ with respect a fixed but arbitrary χ_1 outside some finite set of exceptional characters χ_1 . The short exact sequences

$$0 \rightarrow \mathcal{H}^n(R\bar{f}_*(\bar{K}_\chi))/\pi^r \rightarrow \mathcal{H}^n(R\bar{f}_*(\bar{K}_\chi \otimes^\mathbb{L} \mathfrak{o}/\pi^r \mathfrak{o})) \rightarrow \mathcal{H}^{n+1}(R\bar{f}_*(\bar{K}_\chi))[\pi^r] \rightarrow 0$$

therefore imply that the Q -adic cohomology groups $\mathcal{H}^n(R\bar{f}_*(\bar{K}_\chi))$ vanish for almost all characters χ_2 (with respect to the fixed χ_1). Passing to the algebraic closure Λ of Q we can apply the decomposition theorem and obtain $\mathcal{H}^k(R\bar{f}_*(\bar{K}_\chi)) = \bigoplus_{i+j=k} H^i(\bar{B}, {}^p H^j(R\bar{p}_*(\bar{K}_\chi)))$, and hence the Λ -adic perverse sheaves ${}^p H^j(R\bar{p}_*(\bar{K}_\chi))$ are acyclic for almost all χ_2 (for fixed χ_1). Since the perverse sheaves ${}^p H^j(R\bar{p}_*(\bar{K}_\chi))$ are pure Λ -adic Weil sheaves on \bar{B} , by [W3, lemma 13] then all ${}^p H^j(R\bar{p}_*(\bar{K}_\chi))$ are translation invariant perverse sheaves on \bar{B} . Hence $L = R\bar{p}_*(\bar{K}_\chi) = \bigoplus L_i[-i]$ for certain translation invariant perverse sheaves L_i . Now $L * L \cong H^\bullet(\delta_B) \cdot L \oplus R\bar{p}_*(T_\chi) = H^\bullet(\delta_B) \cdot L$ for almost all χ_1 implies $L = L_0$, since $L_i[-i] * L_i[-i]$ contains nontrivial perverse sheaves in degree $-2i - \dim(B) = -2i - \dim(A)$ for any translation invariant nontrivial perverse sheaf L_i on \bar{B} . Thus for fixed χ_1 (for almost all χ_1) the direct image $R\bar{p}_*(K_\chi)$ itself is a translation invariant perverse sheaf on \bar{B} . Up to a sign $(-1)^{\dim(B)}$ its rank is $\chi(L) = \chi(\bar{K}_\chi|_{\bar{p}^{-1}(0)}) = \chi(\bar{K}|_{\bar{p}^{-1}(0)}) = \chi(R\bar{p}_*(\bar{K})) = (-1)^{\dim(B)} \cdot r$. Using that K and L satisfy $K^\vee \cong K, L \cong L^\vee$ and $L * L \cong H^\bullet(\delta_B) \cdot L$ and $\bar{K}_\chi * \bar{K}_\chi \cong H^\bullet(\delta_B) \oplus \bar{T}_\chi$, which follows from the commutative diagram (*) above, hence by [W3, appendix] we get

$$M^d(b) \cong \delta_0 \quad \text{for } b = 0.$$

In particular the generic rank r of L therefore is one. Hence L is isomorphic to a direct sum of pairwise non-isomorphic translation invariant perverse sheaves of generic rank one for almost all torsion characters χ_1 of $\pi_1(\bar{A}, 0)$ (as explained at the beginning of this section). \square

From the last theorem we get

Proposition 6. *$F(X) = 0$ for complex abelian varieties X isogenous to $A_1 \times A_2$ with simple factors A_1, A_2 of dimension $\dim(A_1) = \dim(A_2)$.*

Proof. Assume there exists $K \in F(X)$. By theorem 5 we conclude that $M = K|_{p^{-1}(0)}$ has Euler characteristic one. This holds also for K replaced by K_χ (for all χ). Hence except for finitely many χ from $H^\bullet(A, M) \cong Rp_*(K_\chi)_0$ and the fact that Euler characteristics of perverse sheaves are nonnegative [FK] we get that the perverse sheaf M^d has Euler characteristic one and M^0, \dots, M^{d-1} are acyclic (using lemma 5). Then all Jordan-Hölder constituents of M^d are acyclic except for one, the perverse sheaf δ_0 (lemma 5) arising from the stalk spectral sequence analogous to lemma 2. In fact this is also clear from an abstract point of view; the constituent with Euler characteristic one is invertible in the Tannakian sense and therefore is a skyscraper sheaf [KrW, prop.21b)]. This Tannakian argument carries over to all fibers $F_b = p^{-1}(b)$ and defines a unique perverse skyscraper Jordan-Hölder constituent in $M(b) = K_\chi|_{F_b}$; in fact a perverse quotient sheaf of $M^d(b)$. Hence for every $b \in B$ this defines a point $x \in X$ with $p(x) = b$. The supports of these skyscraper sheaves define a constructible set; its closure S defines a birational morphism $p : S \rightarrow B$. On an open dense subset $U \subset S$ the morphism p defines an isomorphism onto a dense open subset $V \subset B$. By Milne [M, cor.3.6] the morphism $V \rightarrow U \rightarrow X$ extends to a homomorphism $s : B \rightarrow X$ (up to a translation).

For generic $b \in B$ we have an epimorphism of perverse sheaves

$$M^d(b) \longrightarrow \delta_{s(b)}$$

so that $\delta_{s(b)}$ is the maximal perverse quotient of $M(b)$ with generic support in a subvariety of dimension zero ([KW, lemma III.4.3]). The kernel is acyclic and it is nontrivial! [If it were trivial for generic b , then $\mathcal{H}^{-d}(M^d)$ vanishes. This implies $\mathcal{H}^{-2d}(K|_{F_b}) = 0$ for generic b . Hence $\mathcal{H}^{-\dim(X)}(K)$ vanishes at the generic point of X . In fact the support is contained in $s(B)$. But this is impossible, since then K has support in a simple abelian subvariety and therefore is in $F(X)$.] For generic χ the same conclusion also holds when K is replaced by K_χ . Thus we always find acyclic nontrivial perverse subsheaves of M^d , which are therefore A -invariant. In particular this implies $\text{supp}(K) = X$ (so in fact it would have not been necessary to suppose this). Now we apply verbatim the arguments of step 3) and step 5) of the proof of proposition 2 and conclude as in proposition 2 that the irreducible perverse sheaf K must be translation invariant with respect to A ; indeed $\text{Kern}(p) = A$ is simple and $\text{supp}(K) = X$. A contradiction. \square

9 Appendix

In this section we discuss the situation of abelian varieties over base fields of positive characteristic in the simplest case of abelian varieties isogenous to a product of elliptic curves defined over a finite field.

Let X be a complex elliptic curve and $S \subset X$ a finite set of points. For an irreducible representation $\rho : \pi_1(X \setminus S) \rightarrow Gl(r, \mathbb{C})$ with corresponding local system E on $U = X \setminus S$ let $K = j_*(E)[1]$ denote the associated irreducible perverse sheaf on X defined by the inclusion $j : U \hookrightarrow X$. If E is not trivial, the Euler characteristic $\chi(K) = -\chi(E)$ is equal to $h^1(X, E) = \dim(H^1(X, E))$ and $\chi(E)$ is $(2 - 2g(X))r - \sum_{x \in S} \text{def}_x(j_*(E))$, where $\text{def}_x(j_*(E)) := r - \dim(j_*(E)_x)$. Hence

$$\chi(K) = \sum_{x \in S} \text{def}_x(j_*(E)) .$$

Notice that $\dim(j_*(E)_x)$ coincides with the dimension of the fixed space of $\rho(F_x)$ acting on the representation space $V = \mathbb{C}^r$ of ρ , where F_x denotes a generator of the inertia group of a point over x in the absolute Galois group of the field of meromorphic functions on X . For an elliptic curve X_0 defined over a finite field κ and a smooth etale Λ -adic sheaf E_0 on some Zariski open dense subset $j_0 : U_0 \hookrightarrow X_0$ one similarly defines $K = j_*(E)[1]$, where E, j, U, X and $S = X \setminus U$ are obtained by extending the base field to the algebraic closure $\bar{\kappa}$ of κ . More generally let \mathcal{E} be a constructible Λ -adic sheaf on X with $\mathcal{E}|_U = E$. Then for $K = \mathcal{E}[1]$ the formula of Grothendieck-Ogg-Shafarevic gives

$$\chi(K) = \sum_{x \in S} \text{def}_x(\mathcal{E}) + \sum_{x \in S} \text{swan}_x(E) .$$

The coefficients $\text{swan}_x(E)$ are nonnegative integers and $\text{swan}_x(E)$ vanishes iff the Λ -adic representation ρ of the Galois group of the function field of X_0 , attached to E_0 , is tamely ramified at x . Similarly $\text{def}_x(\mathcal{E})$, defined by $\dim_{\Lambda}(\mathcal{E}_{\bar{\eta}}) - \dim_{\Lambda}(\mathcal{E}_{\bar{x}})$ for geometric points \bar{x} over x and $\bar{\eta}$ over the generic point η of X , vanishes for $\mathcal{E} = j_*(E)$ iff E extends to a smooth etale sheaf over x (see [BBD, 6.1.(b)], p.159). In particular $\text{def}_x(j_*(E)) = 0$ implies $\text{swan}_x(E) = 0$.

Lemma 11. *For all perverse sheaves K on an elliptic curve X defined over \mathbb{C} or a finite field one has $\chi(K) \geq 0$, and $\chi(K) = 0$ for an irreducible perverse sheaf K implies $K \cong \delta_X^{\psi}$ for some character ψ of $\pi_1(X)$, i.e. K is translation-invariant.*

Hence the generic relative vanishing assertion of lemma 1 is also valid for quotient maps by elliptic curves in the case of positive characteristic. From this, by using induction on the dimension, we get the following

Proposition 7. *Let X_0 be an abelian variety isogenous to a product of elliptic curves over a finite field κ and let K_0 be a Λ -adic irreducible perverse sheaf on X_0 also defined over κ . Then $\chi(K) \geq 0$ and $\chi(K) = 0$ implies that K is translation invariant with respect to some elliptic curve in X .*

Proof of the lemma. It suffices to consider the case of an irreducible perverse sheaf. The case of skyscraper sheaves is trivial, so the assertion $\chi(K) \geq 0$ immediately follows from the Grothendieck-Ogg-Shafarevic formula in the case $K = j_*(E)[1]$ and $\chi(K) = 0$ implies $\text{swan}_x(E) = 0$ and $\text{def}_x(j_*(E)) = 0$ for all $x \in X$. Hence E extends to a smooth irreducible representation of $\pi_1(X)$, and therefore K is translation invariant. \square

The following lemma either holds for (X, K) defined over the field of complex numbers or obtained by scalar extension from some (X_0, K_0) defined over a finite field.

Lemma 12. *An irreducible perverse sheaf K on an elliptic curve X with $\chi(K) = 1$ is a skyscraper perverse sheaf concentrated at some point x of X .*

Proof. If the claim were false, then $K \cong j_*(E)[1]$ holds for some open dense subset $j : U \hookrightarrow X$ and some smooth etale Λ -adic sheaf E on U . The assumption $\chi(K) = 1$ implies $U = E \setminus \{x\}$ for some closed point $x \in E$ without restriction of generality. Furthermore the ramification group at x acts tamely on the Λ -adic representation defined by E . Indeed $\chi(K) = \sum_{x \in S} \text{swan}_x(E) + \sum_{x \in S} \text{def}_x(E)$. Since $\text{swan}_x(E)_x$ are integers ≥ 0 , the assumption $\chi(K) = 1$ implies $\text{swan}_x(E) = 0$ for all x and $\text{def}_x(j_*(E)) \neq 0$ for a single point $x \in X$ where $\text{def}_x(j_*(E)) = 1$. For this recall that $\text{def}_x(j_*(E)) = 0$ implies $\text{swan}_x(E) = 0$. So E defines an irreducible Galois representation tamely ramified at x and unramified outside x . The Galois representation can not be unramified on X , since then it would be an irreducible representation of $\pi_1(X)$ and hence $\chi(K) = 0$. Furthermore $E \otimes \chi$ is irreducible and ramified at x for any twist by a character χ of $\pi_1(X)$. Therefore for all characters χ of $\pi_1(X)$

$$H^\bullet(X, K_\chi) = H^0(X, K_\chi) \cong \Lambda$$

holds. Notice $\chi(K^\vee) = \chi(K) = 1$ and hence $\chi(K * K^\vee) = \chi(K)\chi(K^\vee) = 1$. For some semisimple sheaf complex \tilde{K} we have

$$K * K^\vee = \delta_0 \oplus \tilde{K}.$$

We claim that \tilde{K} is a perverse sheaf on X . Otherwise $\mathcal{H}^i(K * K^\vee)_y \neq 0$ holds for some $i > 0$ (and some y in X) or for some $i \leq -2$ (and almost all $y \in X$) as a consequence of the hard Lefschetz theorem. In the latter case the support of $\mathcal{H}^{\geq 0}(K * K^\vee)$ will have dimension one. Since X is a curve, $\mathcal{H}^i(K * K^\vee)_y = 0$ holds for $i > 0$ and $i < -2$. Furthermore $\mathcal{H}^0(K * K^\vee)_y = H^2(X, T_{-y}^*(j_*(E)) \otimes (j_*(D(E))))$ using $D(j_*(E)[1]) = j_*D(E)[1]$, where $D(E)$ is the Verdier dual of E on U . Therefore by excision

$$\mathcal{H}^0(K * K^\vee)_y \cong H_c^2(U, T_{-y}^*(j_*(E))|_U \otimes D(E)) .$$

Since E and $D(E)$ are irreducible, this cohomology group vanishes unless $T_{-y}^*(E)$ and E define isomorphic Galois representations. The irreducible Galois representation defined by E ramifies at x (unless E is unramified which is excluded by $\chi(K) = 1$), whereas $T_{-y}^*(E)$ ramifies at x if and only if $y = 0$. Hence $\mathcal{H}^0(K * K^\vee)$ is a skyscraper sheaf concentrated at $y = 0$. This forces \tilde{K} to be a semisimple perverse sheaf on X with the property

$$\chi(\tilde{K}) = 0 .$$

Furthermore

$$\mathcal{H}^{-1}(\tilde{K})_y = \mathcal{H}^{-1}(K * K^\vee)_y \cong H^1(X, T_{-y}^*(j_*E) \otimes j_*(D(E))) .$$

For fixed $y \in X$ the constructible étale Λ -adic sheaf $T_{-y}^*(j_*E) \otimes j_*(D(E))$ is smooth on $X \setminus \{x, x+y\}$ and defines a Galois representation that is (tamely) ramified at the points x and $x+y$. Since $H^i(X, T_{-y}^*(j_*E) \otimes j_*(D(E))) = 0$ for $i \neq 1$ and $y \neq 0$, for $y \neq 0$ this implies

$$\begin{aligned} h^1(X, T_{-y}^*(j_*E) \otimes j_*(D(E))) &= -\chi(T_{-y}^*(j_*E) \otimes j_*(D(E))) \\ &= \text{def}_x(T_{-y}^*(j_*E) \otimes j_*(D(E))) + \text{def}_{x+y}(T_{-y}^*(j_*E) \otimes j_*(D(E))) \\ &= r(T_{-y}^*(E)) \cdot \text{def}_x(j_*(D(E))) + \text{def}_{x+y}(T_{-y}^*(j_*(E))) \cdot r(D(E)) \\ &= 2r(E) , \end{aligned}$$

where $r = r(E) = r(T_{-y}^*(E)) = r(D(E))$ are the dimensions of the associated Galois representations. Notice $\text{def}_x(j_*(D(E))) = \text{def}_x(j_*(E))$, since $\chi(K) = \chi(K^\vee) = \chi(D(K))$. In other words

$$\dim_\Lambda(\mathcal{H}^{-1}(\tilde{K})_y) = 2r$$

for all $y \neq 0$. By lemma 11 this implies for the semisimple perverse sheaf \tilde{K} the translation invariance so that

$$\tilde{K} \cong \bigoplus_{i=1}^{2r} \delta_X^{\psi_i} , \quad r = r(E) > 0$$

holds for certain characters ψ_i of $\pi_1(X)$. But then $\tilde{K}_\chi = \delta_X \oplus \dots$ for $\chi = \psi_i^{-1}$. This shows $H^\bullet(X, \tilde{K}_\chi) \neq H^0(X, \tilde{K}_\chi)$. Since $K_\chi * K_\chi^\vee = (K * K^\vee)_\chi = \delta_0 \oplus \tilde{K}_\chi$, this implies $H^\bullet(X, K_\chi * K_\chi^\vee) \neq H^0(X, K_\chi * K_\chi^\vee)$ and hence $H^\bullet(X, K_\chi) \neq H^0(X, K_\chi)$ by the Künneth formula and Poincaré duality. This gives a contradiction and proves our claim. \square

Lemma 13. *Let K be an irreducible perverse sheaf on a complex abelian variety X with Euler characteristic $\chi(K) = 1$. Then K is isomorphic to a perverse skyscraper sheaf concentrated at some point $x \in X$.*

Proof. We are free to replace K by a twist with some character χ . For simplicity we then still write K instead of K_χ . For the proof we use induction on the number of simple factors of X (up to isogeny). Let A be a simple abelian subvariety of X and $f : X \rightarrow B = X/A$ be the quotient map. Then $Rf_*(K)$ satisfies $\chi(Rf_*(K)) = 1$ and for a suitable choice of the twist χ we may assume $Rf_*(K)$ to be perverse by lemma 1. Hence by induction and the decomposition theorem we get $Rf_*(K) = \delta_b \oplus \bigoplus_i T_i$ for irreducible perverse sheaves with $\chi(T_i) = 0$ and a skyscraper sheaf δ_b concentrated at some point b (we may assume $b = 0$ by a translation). Therefore $\mathcal{H}^0(Rf_*(K))_b \cong \Lambda$, hence $H^0(A, M) \cong \Lambda$ for $M = K|_A$. Since we replaced K by some generic twist, we may assume that the perverse sheaf $M^0 = {}^p H^0(M)$ on A satisfies $H^\bullet(A, M^0) = H^0(A, M^0)$ and $H^0(A, M^0) = H^0(A, M) = \Lambda$ using the remark of section 1. Therefore the irreducible perverse Jordan-Hölder constituents P of the perverse sheaf M^0 satisfy $\chi(P) = 0$ except for one of them, where $\chi(P) = 1$ holds instead. Since A is simple, $\chi(P) = 0$ implies that P is translation invariant under A by [KrW, prop. 21a], and therefore $\mathcal{H}^{-i}(P) = 0$ holds for $i \neq \dim(A)$. For a short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ of perverse sheaves on A we have an exact sequence of stalk cohomology sheaves at the point a

$$\mathcal{H}^{-1}(W)_a \rightarrow \mathcal{H}^0(U)_a \rightarrow \mathcal{H}^0(V)_a \rightarrow \mathcal{H}^0(W)_a \rightarrow 0.$$

Therefore $\mathcal{H}^0(W)_a \neq 0 \implies \mathcal{H}^0(V)_a \neq 0$. Similarly $\mathcal{H}^{-1}(W)_a = 0$ and $\mathcal{H}^0(U)_a \neq 0$ implies $\mathcal{H}^0(V)_a = 0$. Hence in the case $\dim(A) \geq 2$ we inductively get $\mathcal{H}^0(M^0)_a \cong \mathcal{H}^0(P)_a$ for the unique constituent P with $\chi(P) = 1$. By [KrW, prop. 21b] we know $P \cong \delta_a$ for some point $a \in A$ (we may assume $a = 0$ by a translation of K). This implies $\mathcal{H}^0(M^0)_a \cong \Lambda$. However $\mathcal{H}^0(M)_a = \mathcal{H}^0(M^0)_a$, since $M \in {}^p D^{\leq 0}(A, \Lambda)$. Therefore $\mathcal{H}^0(K)_0 = \mathcal{H}^0(M)_a \cong \Lambda$ holds for the irreducible perverse sheaf K on X . This implies $K = \delta_0$. So it only remains to prove the assertion in the cases where X is isogenous to a product of elliptic curves. Here the same argument can be used as in the case $\dim(A) > 1$, provided we know that the unique irreducible perverse constituent P of M^0 with $\chi(P) = 1$ appears as a quotient perverse sheaf

of the perverse sheaf M^0 on A . To assure this we use a weight argument. This may not be applicable to the pair (X, K) itself. However by the argument of [D] it suffices to prove the statement in all situations where (X, K) is obtained by scalar extension to $\bar{\kappa}$ from a pair (X_0, K_0) defined over a finite field κ . So it suffices to prove the next

Lemma 14. *Let X_0 be an abelian variety defined isogenous to a product of elliptic curves defined over a finite field κ and let K_0 be a Λ -adic irreducible perverse sheaf on X_0 also defined over κ . Then $\chi(K) = 1$ implies that K is a perverse skyscraper sheaf concentrated at some closed point of X .*

Proof. We may assume that K is pure of weight 0. Then δ_b , as constructed above, and hence $H^0(A, M^0)$ are pure of weight zero. Therefore δ_a is pure of weight zero. On the other hand $w(M^0) \leq w(M) \leq w(K) \leq 0$. Hence by the decomposition theorem for pure perverse sheaves on X and the existence of the weight filtration, this implies that δ_a is a quotient perverse sheaf of M^0 . With this additional information we can argue as above in the case $\dim(A) > 1$ to reduce the statement to the case of a single elliptic curve X already discussed in lemma 12.

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